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NOLTR 63-8

A COMPARISON OF NUMERICAL SCHEMES TO CALCULATE THE SOLUTIONS OF A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION WITH SHOCKS

NOL

8 APRIL 1963

UNITED STATES NAVAL ORDNANCE LABORATORY, WHITE OAK, MARYLAND

NOLTR 63-8

Mathematics Department Report M-33

A COMPARISON OF NUMERICAL SCHEMES TO CALCULATE THE SOLUTIONS OF A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION WITH SHOCKS

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ABSTRACT: In a simple problem with shocks and rarefactions for the equation

$$u_t + (\frac{1}{2} u^2 + u)_x = 0 ,$$

six alternative calculation procedures have been tested as to cost and accuracy. Best results were given by the least elaborate method with the finest mesh and by the most elaborate method with the coarsest mesh.

U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

This report is a study of the relative accuracy of six alternative methods of calculating a solution involving shocks and a rarefaction wave for the equation

$$u_t + (\frac{1}{2} u^2 + u)_x = 0 .$$

The equation is equivalent to that proposed by Burgers as a simplified model for shock problems in fluid dynamics. The results are believed to be suggestive as to calculations in such problems.

This work was carried out under NOL Task No. FR-30.

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By direction

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FIGURE

A COMPARISON OF NUMERICAL SCHEMES TO CALCULATE THE SOLUTIONS OF A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION WITH SHOCKS

INTRODUCTION

A simple problem with shocks and rarefactions for the equation

$$(1) \qquad u_t + (\tfrac{1}{2}u^2 + u)_x = 0$$

has been subjected to calculation by various procedures and all the results compared with the exact solution. The methods thus tested are

(1) Lax's original centered difference method [1],

(2) a left difference method without viscosity [2],

(3) the viscosity method of Lax and Wendroff [3]

(3.0) without added artificial viscosity,

(3.1) with added artificial viscosity,

(4) a modification of the Lax-Wendroff method for which convergence has been proved [4]

(4.0) without added artificial viscosity,

(4.1) with added artificial viscosity.

The programming of these schemes for an IBM 704 machine was performed mainly by W. Parr, with the assistance of Mrs. S. Madigosky, for whose patience and intelligent care I wish to express my gratitude and thanks.

Only calculations of roughly the same cost are compared. A multiplication or division costing about the same as ten additions, the measure of relative cost has been taken as

$$C = \frac{10\mu + \alpha}{10{,}000\ hk},$$

where h and k are the horizontal and the vertical distances, respectively, between consecutive grid points (thus, $1/hk$ measures the total number of calculations performed in unit area of the x,t-plane), μ denotes the number of multiplications and divisions performed at a typical grid point, and α the number of other operations at that point.

The problem treated is that of finding a generalized solution [1] of (1) with initial data prescribed as

$$u(x,0) = \begin{cases} 0 & \text{for } x < 0 \\ 2 & \text{for } 0 < x < 0.3 \\ 1 & \text{for } 0.3 < x < 0.9 \\ 0 & \text{for } x > .9 \end{cases}$$

This problem is easily solved explicitly with a rarefaction wave centered at $(0,0)$ and shocks issuing from $(.3,0)$ and $(.9,0)$, each shock being a line of slope

$$\frac{dx}{dt} = 1 + \frac{u_+ + u_-}{2} ,$$

where u_+ and u_- are the limiting values of the solution at the right and at the left of the shock, respectively. In the interval $0 \leq t \leq 0.6$, in particular, the solution of the problem is given by

$$u(x,t) = \begin{cases} 0 & \text{for } x \leq t \\ \frac{x}{t} - 1 & \text{for } t \leq x \leq 3t \\ 2 & \text{for } 3t < x < .3 + \frac{5}{2}t \\ 1 & \text{for } .3 + \frac{5}{2}t < x < .9 + \frac{3}{2}t \\ 0 & \text{for } x > .9 + \frac{3}{2}t \end{cases}$$

(See Figure 1.) This solution is known to be unique.[2]

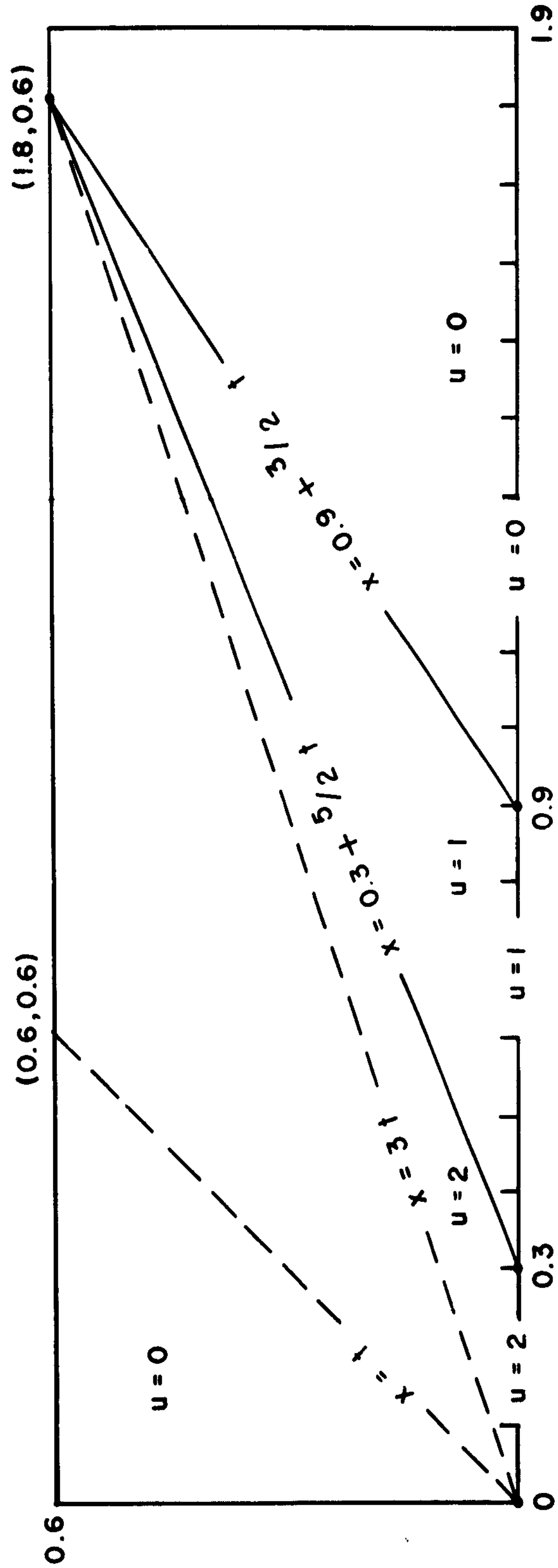

FIG. I EXACT SOLUTION

Note that, in this problem, $0 \le u \le M$, where

$$M = 2 .$$

The same bounds, by a priori reasoning, applies to the calculated values of u in the first, second, and fourth schemes, and for present purposes will be assumed in the third. Note also that all schemes considered, except the third, have been proved to render approximations to the solution which are as close as desired, provided h and k are sufficiently small and the ratio $\theta = k/h$ appropriately bounded.

THE CALCULATION SCHEMES TESTED

The notation in the following is that of [4]: v_{ij} denotes the value of u, as calculated according to the scheme in question, at the grid point $x = ih$, $t = jk$; $w_{ij} = (v_{i+1,j} - v_{ij})/h$ and $W_{ij} = hw_{ij} = v_{i+1,j} - v_{ij}$; $F(u) = u^2/2 + u$; $F_{ij} = F(v_{ij})$, $F'_{ij} = F'(v_{ij})$. Also, as above, $\theta = k/h$.

1. Lax's original centered difference method. Lax's original scheme was based on the difference equations

$$v_{i,j+1} = \frac{1}{2}(v_{i+1,j} + v_{i-1,j}) - \frac{\theta}{2}(F_{i+1,j} - F_{i-1,j}) .$$

According to N. D. Vvedenskaya [6] (Theorem 1), the scheme converges when $\theta \max|F'(v)| \le 1$, or, in our case, when $\theta \le 1/(1 + M) = 1/3$.

2. A left difference method without viscosity. In this scheme,

$$v_{i,j+1} = v_{ij} - \theta(F_{ij} - F_{i-1,j}) .$$

The asymmetry of the scheme is permitted because of the non-negativity of F'. According to a remark at the end of the appendix below, convergence occurs when $\theta \leq 1/4$.

3. <u>The viscosity method of Lax and Wendroff.</u> In this method,

$$v_{i,j+1} = v_{ij} - \frac{\theta}{2}(F_{i+1,j} - F_{i-1,j}) + \frac{\theta^2}{4}[(F'^{\,2}_{ij} + F'^{\,2}_{i+1,j})(v_{i+1,j} - v_{ij}) -$$

$$- (F'^{\,2}_{i-1,j} + F'^{\,2}_{ij})(v_{ij} - v_{i-1,j})] + \theta(Q_{i-1,j} - Q_{ij}) ,$$

where $Q_{ij} = -B|F'_{ij} - F'_{i+1,j}|\,(v_{i+1,j} - v_{ij})$, B being a non-negative constant.here taken as zero in method 3.0 and as 1/4 in method 3.1. Lax and Wendroff [3, p. 227] expect the scheme should be stable, in the case $B = 1/4$, when $\theta \max|F'_{ij}| \leq .78$, thus when $\theta \leq .78/(1+M) = .26$. Stability in the case $B = 0$ by their formula demands merely $\theta \leq 1/3$. (The B in this paper is one fourth the quantity Lax and Wendroff call B.)

4. <u>A modified Lax-Wendroff scheme.</u> The calculation scheme here considered is

$$v_{i,j+1} = v_{ij} - \theta(F_{ij} - F_{i-1,j}) + \frac{\theta^2}{2}[F'^{\,2}_{ij}(v_{i+1,j} - v_{ij}) - F'^{\,2}_{i-1,j}(v_{ij} - v_{i-1,j})] + \theta(Q_{i-1,j} - Q_{ij}),$$

the constant B entering Q_{ij} being taken as zero in Method (4.0) and as 1/4 in Method (4.1). For $B \geq 0$, it is proved in [4] that this method converges if θ is sufficiently small, but larger possible choices for θ than emerge from that discussion are determined in the appendix below; these are

$$\theta = .16 \text{ for } B = 0 \quad \text{(Method (4.0))}$$
$$= .156 \text{ for } B = 1/4 \text{ (Method (4.1))}.$$

Two cost levels ($C = 8$ and $C = 32$) were arbitrarily fixed for all the above schemes and values of θ selected near their permitted maxima. Then mesh widths h and k were obtained to accord in each case with the predetermined C and θ. The values of the parameters thus entering are displayed in Table 1.

MESH WIDTHS, MESH WIDTH RATIOS, and INDICES OF COST

	1	2	3.0	3.1	4.0	1	2	3.0	3.1	4.0	4.1
h	.05148	.05477	.07282	.09682	.09478	.02575	.02739	.03641	.04841	.04739	.05993
k	.01699	.01369	.02403	.02517	.01516	.008495	.006845	.01202	.01258	.007582	.008030
θ	.3300	.2499	.3300	.2600	.1599	.3299	.2499	.3301	.2599	.1600	.1340
θ_{per}	1/3	.25	1/3	.26	.16	1/3	.25	1/3	.26	.16	.156
R	7.0	6.0	14.0	19.5	11.5	7.0	6.0	14.0	19.5	11.5	15.4
C	8.003	8.002	8.001	8.002	8.004	32.00	32.00	31.99	32.02	32.01	32.00

h = mesh width in direction of x-axis
k = mesh width in direction of t-axis
θ = k/h
θ_{per} = maximum permissible value of θ from a priori considerations
R = index of relative cost = $\mu + \alpha/10$, μ denoting the number of multiplications and divisions performed at a typical grid point and α the number of other operations at that point
C = index of total cost = $(R/hk)\cdot 10^{-3}$.

Table 1

CONCLUSIONS

The results of the calculations by each method except 4.1 were tabulated after some thinning for the rectangle

$$R: \quad \begin{array}{l} 0 < x \leq 1.8 \\ 0 < t \leq .6 \end{array} ,$$

the case $C = 32$ being presented here in Tables 4 - 8. The mean absolute error and root mean square error over all mesh points in R and, separately, in its top boundary line L are given for all calculations in Table 2. These results suggest the following observations, at least with respect to the example studied:

1. The use of artificial quadratic viscosity $(B \neq 0)$ in Method 3.1 did not clearly reduce the average error and in Method 4.1 greatly increased it. Other values of B and lower mesh width ratios might, however, have led to better results.

2. Lax-Wendroff viscosity worked well in the method of centered differences (3.0) for which it had been proposed, but disappointingly in the left-difference scheme (4.0). Perhaps the latter method would have performed better, however, at a higher cost level at which the x-axis would be more finely subdivided.

3. Quadrupling the cost less than halved the average error.

4. Methods 2, 3.0, and 3.1 worked best: at identical costs, closely comparable mean errors resulted from the simple scheme and refined mesh of Method 2 and the refined schemes and coarser mesh of Methods 3.0 and 3.1.

Additional conclusions arise from comparing the three most successful methods on a region of relatively great transitions.

AVERAGE ERRORS

Method	C = 8.00				C = 32.00			
	E_R	E_L	S_R	S_L	E_R	E_L	S_R	S_L
1	.20	.18	.31	.28	.12	.11	.20	.18
2	.14	.12	.23	.20	.08	.08	.15	.16
3.0	.16	.11	.29	.22	.08	.07	.18	.23
3.1	.18	.11	.29	.18	.09	.08	.18	.19
4.0	.24	.21	.34	.32	.16	.15	.25	.25
4.1*					.34	.34	.55	.63

R denotes the rectangle $0 < x \leq 1.8$, $0 < t \leq .6$.

L denotes the line segment $0 < x \leq 1.8$, $t = [.6]_k$, where $[s]_k$ is the largest exact multiple of k that does not exceed s.

E_R = average absolute deviation of calculated from exact values of the solution at the lattice points in R.

E_L = average absolute deviation of calculated from exact values of solution at the lattice points on L.

S_R = root mean square deviation at the lattice points in R.

S_L = root mean square deviation at the lattice points on L.

C = index of total cost.

*Better results were obtained by Method 4.1 with B = 1 and a value of θ much $< \theta_{per}$; they still compared poorly with the others.

Table 2

For $C = 32$, in Tables 9, 10, and 11 we present with no thinning in the x-direction such a comparison over the trapezoid

$$T: \begin{array}{l} -.05 + 3t \leq x \leq .95 + 3t/2 \\ .3 \leq t \leq .6 , \end{array}$$

this area covering much of the vicinity of the two shocks and a small part of the rarefaction wave. The mean absolute error and the root mean square error over all the mesh points of T appear in Table 3. From these compilations it seems reasonable to add the following conclusions:

5. By all three methods (2,3.0 and 3.1), calculated values tend to be too low to the left and too high to the right of a shock. These methods thus dull the apparent sharpness of a shock, but Method 2 perhaps to the greatest extent.

6. Method 3.0 seems most prone to occasional wild errors (occurring in this example to the right of the first shock); these errors, however, are rapidly corrected.

7. The average absolute error over T is about the same for the three methods; the mean square error, however, is appreciably less under Methods 2 and 3.1 than under Method 3.0.

Further study of our six methods might consider the effects, at constant cost, of varying the mesh width ratios below their maximum permissible values and also of varying the coefficient of artificial viscosity B.

AVERAGE CALCULATION ERROR

IN

REGION OF RAPID TRANSITIONS

Method	C = 32	
	E_T	S_T
2	.25	.29
3.0	.23	.36
3.1	.23	.29

T denotes the trapezoid $-.05 + 3t \leq x \leq .95 + 3t/2$

$$.3 \leq t \leq .6$$

E_T = average absolute deviation of calculated from exact values of the solution at the lattice points in T .

S_T = root mean square deviation at the lattice points in T .

C = index of total cost

Table 3

TABLE 4 CENTERED DIFFERENCE METHOD OF LAX: METHOD I, C=32.*
CALCULATED VALUES

t \ x	0.	0.08	0.15	0.23	0.31	0.39	0.46	0.54	0.62	0.70	0.77	0.85	0.93	1.00	1.08	1.16	1.24	1.31	1.39	1.47	1.54	1.62	1.70	1.78	1.85	1.93
0.56	0.	0.00	0.00	0.01	0.03	0.06	0.09	0.16	0.22	0.32	0.40	0.52	0.61	0.74	0.84	0.98	1.08	1.23	1.33	1.49	1.57	1.69	0.92	0.27	0.11	0.02
0.47	0.	0.00	0.01	0.02	0.06	0.10	0.18	0.26	0.38	0.48	0.62	0.73	0.90	1.01	1.19	1.30	1.50	1.61	1.82	1.25	0.69	0.40	0.09	0.03	0.00	0.00
0.37	0.	0.00	0.02	0.06	0.11	0.21	0.31	0.47	0.59	0.78	0.92	1.13	1.27	1.51	1.65	1.99	1.30	0.98	0.80	0.35	0.16	0.02	0.01	0.00	0.00	0.00
0.28	0.	0.01	0.05	0.13	0.26	0.40	0.61	0.78	1.05	1.23	1.54	1.73	1.99	1.29	1.05	0.97	0.78	0.54	0.12	0.04	0.00	0.00	0.00	0.	0.	0.
0.19	0.	0.04	0.14	0.35	0.55	0.90	1.15	1.61	2.00	1.99	1.27	1.04	1.00	0.98	0.92	0.49	0.22	0.02	0.00	0.	0.	0.	0.	0.	0.	0.
0.09	0.	0.15	0.56	0.97	2.00	2.00	1.99	1.24	1.00	1.00	1.00	1.00	0.93	0.78	0.13	0.02	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

* $h = .02575$, $k = .008495$, $\theta = .3299$

TABLE 5 LEFT DIFFERENCE METHOD WITHOUT VISCOSITY: METHOD 2, C = 32.*
CALCULATED VALUES

t \ x	0.	0.08	0.16	0.25	0.33	0.41	0.49	0.58	0.66	0.74	0.82	0.90	0.99	1.07	1.15	1.23	1.31	1.40	1.48	1.56	1.64	1.73	1.81	1.89
0.57	0.	0.00	0.00	0.00	0.00	0.02	0.06	0.13	0.23	0.35	0.47	0.59	0.72	0.85	0.98	1.11	1.24	1.37	1.49	1.61	1.63	1.08	0.11	0.00
0.48	0.	0.00	0.00	0.00	0.02	0.07	0.16	0.29	0.42	0.57	0.72	0.87	1.03	1.18	1.33	1.48	1.62	1.70	1.40	0.91	0.21	0.01	0.00	0.00
0.38	0.	0.00	0.00	0.02	0.08	0.20	0.36	0.54	0.72	0.91	1.09	1.28	1.46	1.64	1.77	1.54	1.06	0.94	0.37	0.02	0.00	0.00	0.00	0.00
0.29	0.	0.00	0.02	0.10	0.27	0.48	0.72	0.96	1.20	1.44	1.66	1.84	1.66	1.07	1.00	0.98	0.57	0.04	0.00	0.00	0.00	0.00	0.00	0.00
0.19	0.	0.01	0.13	0.39	0.72	1.06	1.40	1.71	1.92	1.76	1.08	1.00	1.00	0.99	0.77	0.08	0.00	0.00	0.00	0.00	0.00	-0.	-0.	-0.
0.10	0.	0.18	0.70	1.30	1.82	2.00	1.85	1.08	1.00	1.00	1.00	1.00	0.92	0.15	0.00	0.00	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.
0.	0.	2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

* $h = .02739$, $k = .006845$, $\theta = .2499$

TABLE 6 LAX-WENDROFF METHOD WITHOUT ADDED VISCOSITY: METHOD 3.0, C=32.*
CALCULATED VALUES

t \ x	0.	0.07	0.15	0.22	0.29	0.36	0.44	0.51	0.58	0.66	0.73	0.80	0.87	0.95	1.02	1.09	1.17	1.24	1.31	1.38	1.46	1.53	1.60	1.67	1.75	1.82	1.69
0.58	0.	-0.00	0.00	-0.01	0.02	-0.00	-0.05	-0.02	0.07	0.17	0.29	0.40	0.52	0.64	0.76	0.88	1.00	1.12	1.24	1.36	1.48	1.60	1.71	1.87	[illegible]	0.01	0.00
0.48	0.	0.00	-0.01	0.02	-0.01	-0.05	0.01	0.12	0.25	0.38	0.52	0.66	0.80	0.95	1.09	1.23	1.37	1.51	1.66	1.80	2.06	1.23	0.50	0.01	0.00	0.00	0.00
0.38	0.	-0.00	0.02	-0.02	-0.05	[illegible]	0.19	0.35	0.52	0.70	0.87	1.05	1.22	1.40	1.57	1.74	1.92	2.13	0.95	1.16	0.52	0.01	0.00	0.00	0.00	0.00	0.00
0.29	0.	0.01	-0.03	-0.04	0.10	0.30	0.52	0.75	0.98	1.21	1.43	1.66	1.88	2.01	1.58	1.02	0.94	1.15	0.53	0.01	0.00	0.00	0.00	0.00	0.00	-0.	-0.
0.19	0.	-0.03	-0.01	0.22	0.53	0.85	1.18	1.50	1.82	2.00	2.08	1.14	1.00	1.01	0.96	1.14	0.56	0.01	0.00	0.00	0.00	-0.	-0.	-0.	-0.	-0.	-0.
0.10	0.	0.05	0.52	1.09	1.66	2.00	2.00	2.16	1.02	1.00	1.00	1.00	0.99	1.09	0.63	0.01	0.00	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.	-0.
0.	0.	2.00	2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

* $h = .03641$, $k = .01202$, $\theta = .3301$

TABLE 7 LAX-WENDROFF METHOD WITH ADDED VISCOSITY: METHOD 3.1. C=32.*
CALCULATED VALUES

t \ x	0.	0.10	0.19	0.29	0.39	0.48	0.58	0.68	0.77	0.87	0.97	1.07	1.16	1.26	1.36	1.45	1.55	1.65	1.74	1.84	1.94
0.60	0.	-0.00	0.00	0.01	-0.03	-0.02	0.06	0.18	0.32	0.46	0.60	0.75	0.90	1.05	1.20	1.34	1.48	1.70	1.82	0.21	0.00
0.50	0.	-0.00	0.01	-0.03	-0.03	0.07	0.21	0.38	0.55	0.72	0.90	1.07	1.25	1.41	1.60	1.88	1.57	0.45	0.01	0.00	0.00
0.40	0.	0.01	-0.02	-0.03	0.09	0.26	0.46	0.67	0.89	1.11	1.32	1.52	1.79	1.87	1.17	0.89	0.07	0.00	0.00	0.00	0.00
0.30	0.	-0.01	-0.03	0.11	0.34	0.60	0.88	1.16	1.43	1.62	2.01	1.41	0.98	1.13	0.31	0.01	0.00	0.00	0.00	0.00	0.00
0.20	0.	-0.03	0.44	0.47	0.86	1.25	1.62	1.96	1.87	1.06	0.97	1.08	0.77	0.04	0.00	0.00	0.00	0.00	0.	0.	0.
0.10	0.	0.49	0.80	1.46	1.94	2.08	1.26	1.00	1.01	0.98	1.09	0.23	0.00	0.00	0.	0.	0.	0.	0.	0.	0.
0.	0	2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

* $h = .04841$, $k = .01258$, $\theta = .2509$,

TABLE 8 MODIFIED LAX-WENDROFF METHOD WITHOUT ADDED VISCOSITY: METHOD 4.0 C=32.*
CALCULATED VALUES

t \ x	0.	0.09	0.19	0.28	0.38	0.47	0.57	0.66	0.76	0.85	0.95	1.04	1.14	1.23	1.33	1.42	1.52	1.61	1.71	1.80	1.90
0.59	0.	0.00	0.00	0.01	0.04	0.10	0.19	0.30	0.42	0.55	0.68	0.81	0.93	1.05	1.16	1.25	1.29	1.27	1.09	0.64	0.16
0.49	0.	0.00	0.01	0.03	0.10	0.20	0.33	0.48	0.63	0.78	0.93	1.07	1.20	1.29	1.35	1.32	1.16	0.75	0.25	0.04	0.00
0.39	0.	0.00	0.03	0.10	0.22	0.39	0.57	0.75	0.93	1.10	1.25	1.36	1.41	1.36	1.19	0.84	0.37	0.07	0.01	0.00	0.00
0.30	0.	0.01	0.09	0.25	0.46	0.70	0.93	1.14	1.33	1.45	1.48	1.38	1.16	0.89	0.51	0.14	0.02	0.00	0.00	0.00	0.00
0.20	0.	0.07	0.29	0.60	0.92	1.22	1.45	1.58	1.53	1.31	1.08	0.94	0.69	0.24	0.03	0.00	0.00	0.00	0.00	0.00	0.00
0.10	0.	0.34	0.89	1.38	1.69	1.75	1.47	1.11	1.01	0.99	0.87	0.41	0.04	0.00	0.00	0.00	-0.	-0.	-0.	-0.	-0.
0.	0	2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

* $h = .04739$, $k = .007582$, $\theta = .1600$

TABLE 9 LEFT DIFFERENCE EXACT MINUS CA

J \ I	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	
88																
86																
84																
82																
80																
78																
76																
74																
72																
70																
68																
66																.
64															.215	.
62													.203	.230	.258	.
60												.216	.247	.248	.280	.
58										.206	.230	.247	.239	.288	.426	.
56									.220	.247	.232	.233	.300	.464	−.320	−.
54							.212	.235	.245	.218	.229	.319	.507	−.274	−.119	−.
52						.226	.252	.226	.204	.230	.346	−.445	−.230	−.095	−.033	−.
50				.220	.243	.250	.208	.193	.236	.380	−.393	−.190	−.074	−.025	−.006	.
48			.235	.261	.228	.190	.184	.249	.422	−.340	−.153	−.057	−.018	−.004	.004	.
46	.229	.253	.258	.206	.173	.179	.270	.472	−.288	−.121	−.043	−.013	−.003	.003	.014	.

* The dashed line is the right hand boundary of the rarefaction wave: the solid lines intersecting it are the two shocks. The coordinates

$$x = .02739\ (I - 1), \qquad t = .006845\ (J - 1)$$

to which any particular entry corresponds are ascribed to the decimal point for that

1

IOD WITHOUT VISCOSITY: METHOD 2, C=32.
TED VALUES IN REGION OF RAPID TRANSITION*

49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68
																.563	.984	−.533	−.206
														.358	.588	.969	−.587	−.248	
													.387	.619	.050	−.640	−.295		
											.282	.420	.642	−.063	.309	−.348	−.137		
										.307	.458	.659	−.071	.261	−.405	−.172			
								.241	.335	.487	.680	−.074	.217	−.466	−.213				
							.261	.368	.509	−.296	−.073	.178	−.529	−.261					
					.218	.284	.391	.535	−.270	−.069	.144	.409	−.316	−.130					
				.234	.310	.406	.564	−.242	−.063	.115	.349	−.376	−.165						
		.207	.253	.325	.425	.597	−.213	−.056	.091	.293	−.440	−.207							
	.220	.274	.330	.448	−.367	−.185	−.049	.071	.242	−.506	−.255								
36	.282	.339	.477	−.330	−.158	−.041	.055	.197	.428	−.310	−.128								
78	.352	.511	−.291	−.132	−.034	.043	.159	.364	−.371	−.163									
71	.549	−.252	−.109	−.028	.033	.126	.305	−.436	−.204										
10	−.215	−.088	−.022	.025	.099	.251	−.504	−.253											
80	−.071	−.017	.019	.077	.203	.429	−.309	−.126											
56	−.013	.014	.059	.163	.365	−.370	−.161												
10	.011	.045	.128	.304	−.435	−.202													
08	.034	.100	.250	−.303	−.251														
25	.077	.201	.429	−.306	−.123														
59	.160	.363	−.368	−.158															
25	.302	−.434	−.199																

2

TABLE 10 LAX-WENDROFF METHOD WITHOUT ADDED VISCOSITY: METHOD 3.0, C=32. EXACT MINUS CALCULATED VALUES IN REGION OF RAPID TRANSITION *

J \ I	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
50																									-.067	-1.53	-.133
49																								.035	-1.02	-.401	
48																							.050	-.167	-.409	-.079	
47																						.112	[illegible]	-.491	-.234		
46																					.120	.037	[illegible]	.437	-.072		
45																				.123	.021	-1.02	-.010	-.208			
44																			.134	.051	-1.03	-.368	[illegible]	-.072			
43																		.140	.071	-1.03	-.518	.113	-.207				
42																	.145	.090	-.091	-.547	-.190	.509	-.072				
41																.150	.108	-.072	-.589	-.232	.125	-.210					
40															.154	.120	-.063	-.705	-.229	-.166	.505	-.074					
39														.158	.128	-.023	-.379	-.160	-.242	.121	-.213						
38													.162	.134	.022	-1.05	-.156	-.166	-.166	.500	-.075						
37												.166	.140	.052	-1.16	-.249	-.058	-.237	.116	-.216							
36											.170	.147	.067	-.392	-.430	.013	-.158	-.169	.494	-.076							
35										.175	.154	.073	-.165	-.654	.024	-.044	-.237	.111	-.219								
34									.179	.160	.075	-.113	-.865	-.027	.038	-.156	-.172	.490	-.078								
33								.184	.165	.077	-.065	-1.03	-.123	.068	-.042	-.238	.106	-.221									
32							.188	.171	.081	-.033	-1.13	-.248	.052	.040	-.156	-.174	.485	-.079									
31						.194	.177	.086	-.015	-1.23	-.392	.010	.072	-.042	-.238	.101	-.224										
30					.199	.183	.091	-.008	-.177	-.551	-.037	.061	.039	-.155	-.177	.480	-.080										
29				.205	.190	.096	-.006	-.147	-.721	-.030	.028	.071	-.042	-.238	.096	-.227											
28			.211	.197	.101	-.006	-.102	-.889	-.123	-.004	.061	.038	-.155	-.178	.475	-.082											
27		.217	.205	.105	-.006	-.059	-1.03	-.183	-.024	.030	.069	-.042	-.237	.092	-.231												
26	.224	.213	.110	-.005	-.029	-1.13	-.274	-.031	.000	.058	.037	-.154	-.180	.470	-.083												

* The dashed line is the right hand boundary of the rarefaction wave: the solid lines intersecting it are the two shocks. The coordinates

$x = .03641\ (I - 1), \quad t = .01202\ (J - 1)$

to which any particular entry corresponds are ascribed to the decimal point for that entry.

TABLE II LAX-WENDROFF METHOD WITH ADDED VISCOSITY: METHOD 3.1, C=32, B=1/4 EXACT MINUS CALCULATED VALUES IN REGION OF RAPID TRANSITION*

J \ I	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
48																			.130	.106
47																		.021	.172	-.556
46																		.197	-.058	-.238
45																	.081	-.494	-.593	
44																.053	.253	-.068	-.287	
43															.055	.148	-.446	-.649		
42														.066	.115	-.669	-.071	-.357		
41														.090	.222	-.390	.278	-.181		
40													.076	.160	-.570	-.090	-.446			
39												.083	.112	-.672	-.334	.192	-.251			
38											.109	.087	.236	-.459	-.115	.448	-.129			
37											.091	.142	-.544	-.283	.102	-.339				
36										.112	.076	-.659	-.349	-.140	.333	-.186				
35									.145	.061	.202	-.408	-.237	.020	-.445					
34								.179	.091	.089	-.532	-.250	-.157	.217	-.263					
33							.204	.126	.039	-.709	-.280	-.193	-.050	.438	-.139					
32							.188	.046	.131	-.398	-.166	-.163	.111	-.358						
31						.216	.090	.039	-.595	-.171	-.151	-.101	.316	-.201						
30					.236	.149	.016	.207	-.273	-.098	-.157	.019	-.468							
29				.249	.209	.040	.071	-.466	-.088	-.110	-.133	.198	-.283							
28				.257	.089	.008	-.683	-.170	-.046	-.140	-.052	.411	-.152							
27			.273	.147	.001	.141	-.339	-.031	-.072	-.146	.093	-.382								
26		.284	.204	.032	.032	-.549	-.094	-.008	-.115	-.102	.289	-.219								
25	.288	.258	.082	-.014	-.748	-.229	.003	-.037	-.143	.005	-.496									

* The dashed line is the right hand boundary of the rarefaction wave; the solid lines intersecting it are the two shocks. The coordinates

$$x = .04841\ (I-1), \qquad t = .01258\ (J-1)$$

to which any particular entry corresponds are ascribed to the decimal point for that entry.

APPENDIX

Our aim here is to find as large values of θ as possible for which methods 4.0 and 4.1 will converge. Two procedures will be compared, the first that described in [4] and the second a modification of it. The second procedure turns out to be the better and leads to the values actually used above.

Calculation scheme 4 is that of [4] with

$$f = F, \quad g = 0, \quad s = 0, \quad q_{ij} = -\frac{k}{2}(1 + v_{ij})^2 w_{ij} + Q_{ij} ,$$

see pp. 5 and 21 and equation (4.4b).[3] In inequality (4.7), in particular, therefore, as we see at once,

$$b_{ij} = \frac{\theta}{2}(1 + v_{i-1,j})^2 + B|W_{i-1,j}| , \quad c_{ij} = \frac{\theta}{2}(1 + v_{ij})^2 + B|W_{ij}| ,$$

$$d_{ij} = e_{ij} = 0 .$$

To obtain inequality (4.8), it is well to note that

$$Q_{ij} = -B|W_{ij}|W_{ij} = f(W_{ij}) , \qquad \text{where} \quad f(W) = -B|W|W ,$$

and to use the method described for case (4.5c). In the result, we evidently will have

$$B_{ij} = \frac{\theta}{2}(1 + v_{ij})^2 + 2B|W'_{ij}| , \quad C_{ij} = \frac{\theta}{2}(1 + v_{i+1,j})^2 + 2B|W''_{ij}| , \quad H = \max H_{ij} ,$$

$$H_{ij} = 1 + (v_{ij} + v_{i+1,j})/2 , \quad D_{ij} = E_{ij} = E'_{ij} = E''_{ij} = 0 ,$$

where W'_{ij} and W''_{ij} are numbers between W_{ij} and $W_{i-1,j}$, and between W_{ij} and $W_{i+1,j}$, respectively.

Both ways of bounding θ begin with the argument for Theorem 5.1. In this argument, θ is restricted by the single requirement that the coefficient of v_{ij} in the right member of (5.2) be non-negative, and, therefore, that

$$1 - \theta(F'^* + b^* + c^* = ks^*) \geq 0 \quad ,$$

the star in each case indicating a maximum value of the quantity concerned for the various possible values of its indices or arguments. Since $F'^* = 1 + M$, $b^* = c^* = \frac{\theta}{2}(1 + M)^2 + MB$, $s^* = 0$, we have the condition

$$(2) \qquad (1 + M)^2\theta^2 + (1 + M + 2MB)\theta - 1 \leq 0 \; .$$

M being equal to 2, for $B = 0$ we thus have

$$(3)_0 \qquad \theta \leq (5^{1/2} - 1)/6 = .206^+$$

and for $B = 1/4$

$$(3)_1 \qquad \theta \leq \frac{13-2}{9} = .178 \, .$$

In the first of the two ways of bounding, θ now is subjected to an additional restriction arising from the argument for Theorem 5.2. According to this argument, the coefficients

$$p_{ij} = 1 - \theta[1+v_{ij} + \frac{\theta}{2}(1+v_{i,j})^2 + \frac{\theta}{2}(1+v_{i+1,j})^2 + 2B|W'_{ij}| + 2B|W''_{ij}| \;]$$

and

$$a = \frac{1}{2}(1 - 2\theta(1+M))$$

in (5.8) ($b=0$ in this equation) must be made to satisfy the conditions (p.14) $p_{ij} \geq 2/3$ and $a > 0$; hence, we require

$$(1 + M)^2\theta^2 + (1 + M + 4BM)\theta - 1/3 \leq 0 \quad \text{and} \quad \theta < 1/2(1+M) .$$

The first of these inequalities would imply the second. Since $M = 2$, the first inequality is equivalent to

$$(4)_0 \qquad \theta \leq \frac{(7/3)^{1/2} - 1}{6} = .087^+ \quad \text{for} \quad B = 0$$

and to

$$(4)_1 \qquad \frac{\sqrt{37} - 5}{18} = .06 \quad \text{for} \quad B = 1/4$$

It expresses the first of the two alternative bounds we have considered upon θ .

The second method of bounding θ comes from a second method of proof of Theorem 5.2 in which an argument of Vvedenskaya has been applied. We begin with inequality (5.7) which, in the present case, reads

$$w_{i,j+1} \leq [1 - \theta(F'_{ij} + B_{ij} + C_{ij})]w_{ij} + \theta(F'_{ij} + B_{ij})w_{i-1,j} + \theta C_{ij}w_{i+1,j}$$

$$- \frac{k}{2}[\overline{F}''_{ij} - 2\theta(1 + \frac{v_{ij}+v_{i+1,j}}{2})]w_{ij}^{\,2} - \frac{k}{2}\,\overline{\overline{F}}''_{ij}w_{i-1,j}^{\,2} \quad ,$$

single or double bars connoting intermediate values of the arguments as occurring in Taylor's theorem with remainder. The bracketed part of the coefficient of $w_{ij}^{\,2}$ is required to be positive, as previously, but the coefficient of w_{ij} , unlike before, is to be merely non-negative. Hence, θ now is to be such that $\min F'' - 2\theta(1+M) > 0$ and $1 - \theta(F'^{*}+B^{*}+C^{*}) \geq 0$, the star again signifying maxima of the quantities in question.

Since $F'' = 1$, the first of these conditions is

$$(5)_o \qquad \Theta < 1/2(1+M) = \frac{1}{6} = .166^+ .$$

The second, which we write as

$$(1+M)^2 \Theta^2 + (1 + M + 4MB)\Theta - 1 \leq 0 ,$$

reduces to $(3)_o$ for $B = 0$ and to the condition

$$(5)_1 \qquad \Theta \leq \frac{\sqrt{61} - 5}{18} = .156 \qquad \text{for } B = 1/4 .$$

No stronger condition, as we shall now see, arises out of the further argument. Let

$$\tilde{w}_{ij} = \max (w_{i-1,j}, w_{ij}, 0) , \quad N_j = \max (0, \max_i w_{ij}) .$$

From the above inequality, we have

$$(6) \qquad w_{i,j+1} \leq (1 - \Theta C_{ij})\tilde{w}_{ij} + \Theta C_{ij} N_j - \frac{c}{2} k\tilde{w}_{ij}^2 ,$$

where $c = \min F'' - 2\Theta(1+M) = 1 - 2\Theta(1+M)$. (In this type of argument we follow Vvendenskaya.) Since Θ is supposed to satisfy the conditions of the previous paragraph, $c > 0$.

Next, like Vvedenskaya [6], consider the quadratic expression

$$H(y) \equiv H_{ij}(y) \equiv (1 - \Theta C_{ij})y - \frac{c}{2}ky^2 .$$

H will be monotonically increasing, i.e., $H'(y) = 1 - \Theta C_{ij} - kcy \geq 0$, if $y \leq (1 - \Theta C_{ij})/kc$. Hence, in particular

(7) $$H(\tilde{w}_{ij}) \leq H(N_j) \quad ,$$

if, for all i, $N_j \leq (1 - \theta C_{ij})/kc$. In (5.10) we have proved, however, that $N_j \leq \sup_i w_{io} \leq m/h$, m being a bound for v_{io}. (In the present case, $w_{ij} = W_{ij}$ as used on p. 14, and $b = 0$.) Hence, N_j satisfies the desired condition (7), if θ is so fixed that, for all i, $m/h \leq (1-\theta C_{ij})/kc$. Substituting in this for c and also replacing C_{ij} by its upper bound $\frac{\theta}{2}(1 + M)^2 + 2MB$, we arrive at the condition

$$(1+M)(2m - \tfrac{1}{2}(1 + M))\theta^2 - (m + 2MB)\theta + 1 \geq 0 \quad ,$$

which, for $m = M = 2$ becomes

$$\frac{15}{2}\theta^2 - (2 + 4B)\theta + 1 \geq 0$$

For $B = 0$ or $B = 1/4$, the quadratic expression on the left is definite: the inequality is satisfied for every value of θ and does not really constitute a restriction.

We are enabled by (7) to replace $\tilde{w}_{ij}$ in the right member of (6) by N_j and, thus, to obtain

$$w_{i,j+1} \leq N_j - \frac{c}{2}kN_j^{\,2} \quad ,$$

and, therefore,

(8) $$N_{j+1} \leq N_j - \frac{c}{2}kN_j^{\,2} \quad .$$

From this, it is easy to produce an upper bound for N_j completing this alternative proof of Theorem 5.2. We shall show, to be specific, that

(9) $$N_j \leq 2/cjk \ .$$

Set $z(t) = 2/ct$, $z_j = z(jk)$. Since $dz/dt = -cz^2/2$, integrating we obviously have

$$z_{j+1} - z_j = -\frac{c}{2} \int_{jk}^{(j+1)k} z(t)^2 \, dt > -\frac{c}{2} k z_j^2 \ ,$$

or

(10) $$z_{j+1} > H(z_j) \ ,$$

where

$$H(y) \equiv y - cky^2/2 \ .$$

From this we shall prove by induction that

(11) $$z_j \geq N_j \ , \quad j = 1,2,\ldots \ ;$$

these inequalities are equivalent to (9). The first step is to note that $z_1 \geq N_1$, $z_2 \geq N_2$, consequences of the fact discussed in a previous paragraph that $1/ck > m/h \geq N_1, N_2$. Next observe that $H(y)$ monotonically increases for $y \leq 1/ck$ and, furthermore, that $z_j \leq 1/ck$ for $j \geq 2$. Therefore, if $z_j \geq N_j$ for some index $j \geq 2$, for that index $H(z_j) \geq H(N_j)$. From this, (10), and (8), if for a particular index $j \geq 2$ we know $z_j \geq N_j$, we can conclude that

$$z_{j+1} > H(z_j) \geq H(N_j) \geq N_{j+1} \ .$$

The induction for (11) is thus complete.

We note in summary that our first method bounds Θ according to (3) and (4), our second method according to (3) and (5). The second results are obviously better; they permit use of the values

$$\Theta = .166 \quad \text{for } B = 0 \tag{12}$$
$$\quad = .156 \quad \text{for } B = 1/4$$

in the calculations.

Remark: The left difference scheme without viscosity of section 2 is of type (4.4b) with

$$f_{ij} = F_{ij}, \quad g_{ij} = q_{ij} = s_{ij} = 0 .$$

Reasoning like that above concerning Theorem 5.1 shows that, in this situation, $M = m$ if $\Theta \max F' \leq 1$, i.e., if $\Theta \leq 1/(1+M) = 1/3$. The further restriction

$$\Theta \leq 1/2m = 1/4 \tag{13}$$

is easily seen to be required to bound w_{ij} (by the second method) according to Theorem 5.2. To show this, we start from (5.6) which, here, leads to the inequality

$$w_{i,j+1} \leq (1 - \Theta F'_{ij})w_{ij} + \Theta F'_{ij}w_{i-1,j} - \frac{k}{2}(F''_{ij}w_{ij}^2 + F''_{ij}w_{i-1,j}^2)$$

and thus, in previous notation, to

$$w_{i,j+1} \leq \tilde{w}_{ij} - \frac{k}{2}\tilde{w}_{ij}^2$$

since $F'' = 1$ and since the coefficients on the right are all positive because of the first restriction upon θ . The previous argument leading to the desired upper bound for w_{ij} will apply completely, provided we now require $2m/h \leq 1/k$, a restriction which is the equivalent of (13).

FOOTNOTES

1/ By generalized solution we mean essentially a piecewise continuous solution satisfying the pertinent shock condition along discontinuities. More refined concepts of generalized solution are to be found in [2] and the accompanying references.

2/ For uniqueness theorem, cf [5].

3/ This and other numbered equations, theorems, etc., not belonging to the present paper are to be found in [4].

REFERENCES

1. P. D. Lax, Weak solutions of nonlinear hyperbolic equations and their numerical computation, Comm. Pure Appl. Math., 7(1954),159-193

2. A. Douglis, On discontinuous solutions of quasi-linear, first order partial differential equations, NAVORD Report 6775, U. S. Naval Ordnance Lab., White Oak, Md., 1960, pp. 1-75

3. P. D. Lax and B. Wendroff, Systems of Conservation Laws, Comm. Pure Appl. Math., 13(1960), 217-238

4. A. Douglis, On calculating solutions of quasi-linear, first order partial differential equations, NAVWEPS Report 7254, U. S. Naval Ordnance Lab., White Oak, Md., 1960, pp. 1-23

5. O. A. Oleinik, On discontinuous solutions of nonlinear differential equations, Dokl. Akad. Nauk SSSR 109(1956), 1098-1101

6. N. D. Vvedenskaya, The difference method solution of Cauchy's problem for a non-linear equation with discontinuous initial values, Dokl. Akad. Nauk SSSR 111(1956), 517-521

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REPORT NUMBER	63-8	63ØØØ8	CIRCULATION LIMITATION		
REPORT DATE	8 April 1963	Ø463	CIRCULATION LIMITATION OR BIBLIOGRAPHIC		
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DESCRIPTORS	CODES	DESCRIPTORS	CODES	DESCRIPTORS	CODES
Calculating	COMA	Accuracy	ACUR	Programed	PROR
Solutions	SOLG	Method	METD	IBM	IBMA
Equation	EQUA	Mesh	MESH	Computer	COMP
Shock waves	SHWV	Size	SIZE	704	Ø7Ø4
Comparison	CMRI	Fluid	FLUI	Shock	SHOC
Numerical	NUMB	Dynamics	DYNA	Explosion	EXPS
Schemes	PLAG	Lax	LAXZ		
Non-linear	NONI	Difference equation	DIES		
Partial	PARI	Viscosity	VISC		
Differential	DIFE	Wendroff	WEND		
Rarefaction wave	RRFC	Artificial	SYNT		
Cost	COST	Convergence	CNVG		

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Naval Ordnance Laboratory, White Oak, Md.
(NOL technical report 63-8)
A COMPARISON OF NUMERICAL SCHEMES TO CALCULATE THE SOLUTIONS OF A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION WITH SHOCKS (U), by A. Douglis. 8 April 1963. 30p. tables. (Mathematics Dept. report M-33) NOL task FR-30. UNCLASSIFIED

In a simple problem with shocks and rarefactions for the equation

$$u_t + (\frac{1}{2} u^2 + u)_x = 0,$$

six alternative calculation procedures have been tested as to cost and accuracy. Best results were given by the least elaborate method with the finest mesh and by the most elaborate method with the coarsest mesh.

1. Fluid dynamics Equations
2. Non-linear equations
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